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Three-dimensional motion of a vortex filament and its relation to the localized induction hierarchy

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Abstract. Three-dimensional motion of a slender vortex tube, embedded in an inviscid incompressible fluid, is investigated under the *localized induction approximation* for the Euler equations. Using the method of matched asymptotic expansions in a small parameter ϵ, the ratio of core radius to curvature radius, the velocity of a vortex filament is derived to $O(\epsilon^3)$, whereby the influence of elliptical deformation of the core due to the self-induced strain is taken into account. It is found that there is an integrable line in the core whose evolution obeys a summation of the first and third terms of the *localized induction hierarchy.*

PACS. 47.32.Cc Vortex dynamics – 02.30.Ik Integrable systems

1 Introduction

An asymptotic theory that concisely spotlights some qualitative behavior of a curved vortex filament in an inviscid incompressible fluid is the so called '*localized induction approximation (LIA)*' [1]. The filament curve $\boldsymbol{X} = \boldsymbol{X}(s, t)$, expressed as functions of the arclength s and the time t evolves according to

$$\boldsymbol{X}_t = C\kappa \boldsymbol{b}\,; \qquad C = \frac{\Gamma}{4\pi}\log\left(\frac{L}{\sigma_0}\right), \tag{1}$$

where κ is the curvature, $\boldsymbol{b}$ the binormal vector, Γ the circulation, and a subscript denotes a differentiation with respect to the indicated variable. The long and short cut-off lengths L and σ_0 for the Biot-Savart law and thus C are assumed to be constant.

A distinguishing feature is that (1) is a completely integrable evolution equation equivalent to a cubic nonlinear Schrödinger equation (NLS) for the Hasimoto map:

$$\phi(s,t) = \kappa\, \mathrm{e}^{\mathrm{i}\int^s \tau \mathrm{d}s}, \tag{2}$$

a combination of curvature κ and torsion τ [2]. Magri [3] unveiled the bi-Hamiltonian structure behind the integrability of NLS, and manipulated a recursion operator generating successively an infinite sequence of commuting vector fields. Relying on this, Langer and Perline [4] constructed its counterpart for (1). The resulting sequence of integrable vector fields is called the '*localized induction hierarchy (LIH)*'. The first three of them are

$$\boldsymbol{V}^{(1)} = \kappa \boldsymbol{b}\,, \quad \boldsymbol{V}^{(2)} = \frac{1}{2}\kappa^2 \boldsymbol{t} + \kappa_s \boldsymbol{n} + \kappa\tau \boldsymbol{b}\,,$$
$$\boldsymbol{V}^{(3)} = \kappa^2\tau \boldsymbol{t} + (2\kappa_s\tau + \kappa\tau_s)\,\boldsymbol{n} + \left(\kappa\tau^2 - \kappa_{ss} - \frac{1}{2}\kappa^3\right)\boldsymbol{b}\,, \tag{3}$$

where $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$ are the Frenet-Serret vectors. Remarkably, when specialized to a circle with constant curvature $\kappa \neq 0$ and $\tau = 0$, a superposition of $\boldsymbol{V}^{(1)}$ and $\boldsymbol{V}^{(3)}$ coincides with the higher-order formula for traveling speed of a thin axisymmetric vortex ring [5,6].

This unexpected coincidence inspires us to pursue the higher-order velocity of a vortex filament. We note in passing that the Moore-Saffman filament equation [7], valid to $O(\epsilon^2)$, for a vortex filament with axial velocity in the core is reducible to a summation of $\boldsymbol{V}^{(1)}$ and $\boldsymbol{V}^{(2)}$ [8].

Here, we rule out axial flow at leading order, but make an attempt at an extension of matched asymptotic expansions to $O(\epsilon^3)$ under the LIA.

2 Setting of problem

In order to look into the flow field near the core, it is expedient to introduce local coordinates $(\tilde{x}, \tilde{y}, \xi)$, along with local cylindrical coordinates (r, φ, ξ) such that $\tilde{x} = r\cos\varphi$ and $\tilde{y} = r\sin\varphi$, moving with the filament. Here ξ is a parameter along the central curve $\boldsymbol{X}$ of the vortex tube,

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defined so as to satisfy $\dot{\boldsymbol{X}}(\xi,t)\cdot\boldsymbol{t}(\xi,t)=0$. Here a dot stands for a derivative in t with fixing ξ. Given a point $\boldsymbol{x}$ sufficiently close to the core, there corresponds uniquely the nearest point $\boldsymbol{X}(\xi,t)$ on the centerline of filament. Then $\boldsymbol{x}$ is expressed as

$$\boldsymbol{x}=\boldsymbol{X}(\xi,t)+r\cos\varphi\boldsymbol{n}+r\sin\varphi\boldsymbol{b}\,. \tag{4}$$

The coordinates (r,φ,ξ) are converted into orthogonal ones (r,θ,ξ) by adjusting the origin of angle as

$$\theta(\varphi,\xi,t)=\varphi-\int_{s_0}^{s(\xi,t)}\tau(s',t)\,\mathrm{d}s'\,, \tag{5}$$

where $s=s(\xi,t)$ is the arclength along the centerline [7].

We define the relative velocity $\boldsymbol{V}=(u,v,w)$ as functions of r,θ,ξ and t by

$$\boldsymbol{v}=\dot{\boldsymbol{X}}(\xi,t)+u\boldsymbol{e}_r+v\boldsymbol{e}_\theta+w\boldsymbol{t}\,, \tag{6}$$

where $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are the unit vectors in the radial and azimuthal directions respectively. The vorticity $\boldsymbol{\omega}=\nabla\times\boldsymbol{v}$ is calculated through

$$\boldsymbol{\omega}=\omega_r\boldsymbol{e}_r+\omega_\theta\boldsymbol{e}_\theta+\zeta\boldsymbol{t} \tag{7}$$

$$=\left\{\frac{1}{r}\frac{\partial w}{\partial\theta}-\frac{1}{h_3}\frac{\partial v}{\partial\xi}+\frac{\eta}{h_3}\kappa w\sin\varphi-\frac{1}{h_3}\frac{\partial\dot{\boldsymbol{X}}}{\partial\xi}\cdot\boldsymbol{e}_\theta\right\}\boldsymbol{e}_r$$
$$+\left\{-\frac{\partial w}{\partial r}+\frac{1}{h_3}\frac{\partial u}{\partial\xi}+\frac{\eta}{h_3}\kappa w\cos\varphi+\frac{1}{h_3}\frac{\partial\dot{\boldsymbol{X}}}{\partial\xi}\cdot\boldsymbol{e}_r\right\}\boldsymbol{e}_\theta$$
$$+\left\{\frac{1}{r}\frac{\partial}{\partial r}(rv)-\frac{1}{r}\frac{\partial u}{\partial\theta}\right\}\boldsymbol{t}\,, \tag{8}$$

where

$$\eta=|\partial\boldsymbol{X}/\partial\xi|\,,\quad h_3=\eta(1-\kappa r\cos\varphi)\,. \tag{9}$$

We are concerned with a *quasi-steady* motion of a vortex filament. Suppose that the leading-order flow is circulatory motion with prescribed velocity field $v^{(0)}(r)$ as a function only of r. Consistently with the LIA, we may pose the following form for the perturbation solution in a power series in $\epsilon=\sigma_0/R_0$, the ratio of a typical core radius σ_0 to a typical curvature radius R_0:

$$u=\epsilon u^{(1)}+\epsilon^2u^{(2)}+\epsilon^3u^{(3)}+\cdots\,,$$
$$v=v^{(0)}(r)+\epsilon v^{(1)}+\epsilon^2v^{(2)}+\epsilon^3v^{(3)}+\cdots\,,$$
$$w=\epsilon^2w^{(2)}+\cdots\,,\quad \dot{\boldsymbol{X}}=\dot{\boldsymbol{X}}^{(0)}+\epsilon^2\dot{\boldsymbol{X}}^{(2)}+\cdots\,. \tag{10}$$

Inspection from (8) and (10) tells us that

$$\omega_r=\epsilon^2\omega_r^{(2)}+\cdots\,,\quad \omega_\theta=\epsilon^2\omega_\theta^{(2)}+\cdots\,,$$
$$\zeta=\zeta^{(0)}(r)+\epsilon\zeta^{(1)}+\epsilon^2\zeta^{(2)}+\epsilon^3\zeta^{(3)}+\cdots\,. \tag{11}$$

To integrate the Euler equations, it is advantageous to eliminate the pressure at the outset and to deal exclusively with vorticity and vector potential $\boldsymbol{A}$ for the velocity: $\boldsymbol{v}=\nabla\times\boldsymbol{A}$. Introduce a Stokes streamfunction

$$\psi(\boldsymbol{x})=(1-\kappa r\cos\varphi)\boldsymbol{A}(\boldsymbol{x})\cdot\boldsymbol{t}(\xi) \tag{12}$$
$$=\psi^{(0)}(r)+\epsilon\psi^{(1)}+\epsilon^2\psi^{(2)}+\epsilon^3\psi^{(3)}+\cdots\,, \tag{13}$$

for the flow in the plane transversal to $\boldsymbol{t}$.

3 Asymptotic development of Biot-Savart law

This section presents only a brief sketch of how to perform an asymptotic development, valid near the core, of the Biot-Savart law for $\boldsymbol{A}(\boldsymbol{x})$.

The vorticity is dominated by the tangential contribution ζ. We stipulate that $|\zeta|$ decays sufficiently rapidly to zero with distance r from the vortex centerline. The contribution $\boldsymbol{A}_\parallel$ from ζ is

$$\boldsymbol{A}_\parallel(\boldsymbol{x})=\frac{1}{4\pi}\iiint\frac{\zeta(\tilde{x},\tilde{y})\boldsymbol{t}(s)(1-\kappa\tilde{x})}{|\boldsymbol{x}-\boldsymbol{X}-\tilde{x}\boldsymbol{n}-\tilde{y}\boldsymbol{b}|}\,\mathrm{d}\tilde{x}\mathrm{d}\tilde{y}\mathrm{d}s\,. \tag{14}$$

Use of a shift-operator, being adapted from Dyson's technique [5], facilitates to rewrite (14) in a form amenable to a multi-pole expansion as

$$\boldsymbol{A}_\parallel(\boldsymbol{x})=\frac{1}{4\pi}\int\mathrm{d}s\Bigg\{\iint\mathrm{d}\tilde{x}\mathrm{d}\tilde{y}\zeta(\tilde{x},\tilde{y})(1-\kappa\tilde{x})$$
$$\times\exp\left[-\tilde{x}(\boldsymbol{n}\cdot\nabla)-\tilde{y}(\boldsymbol{b}\cdot\nabla)\right]\Bigg\}\frac{\boldsymbol{t}(s)}{|\boldsymbol{x}-\boldsymbol{X}(s)|} \tag{15}$$

$$=\frac{1}{4\pi}\int\mathrm{d}s\Bigg\{\iint\mathrm{d}\tilde{x}\mathrm{d}\tilde{y}\zeta\Big(1-\kappa\tilde{x}-\tilde{x}(\boldsymbol{n}\cdot\nabla)-\tilde{y}(\boldsymbol{b}\cdot\nabla)$$
$$+\frac{1}{2}\left[\tilde{x}^2(\boldsymbol{n}\cdot\nabla)^2+2\tilde{x}\tilde{y}(\boldsymbol{n}\cdot\nabla)(\boldsymbol{b}\cdot\nabla)+\tilde{y}^2(\boldsymbol{b}\cdot\nabla)^2\right]$$
$$+\kappa\tilde{x}^2(\boldsymbol{n}\cdot\nabla)+\kappa\tilde{x}\tilde{y}(\boldsymbol{b}\cdot\nabla)+\cdots\Big)\Bigg\}\frac{\boldsymbol{t}(s)}{|\boldsymbol{x}-\boldsymbol{X}(s)|}\cdot \tag{16}$$

We shall know from the inner expansion in Section 4 the following dependence of ζ on φ:

$$\zeta(\tilde{x},\tilde{y})=\zeta_0+\zeta_{11}\cos\varphi+\zeta_{12}\sin\varphi+\zeta_{21}\cos2\varphi+\cdots\,, \tag{17}$$

where

$$\zeta_0\approx\zeta^{(0)}(r)+\kappa^2\hat{\zeta}_0^{(2)}(r)\,,\quad \zeta_{11}\approx\kappa\hat{\zeta}_{11}^{(1)}(r)+\kappa^3\hat{\zeta}_{11}^{(3)}(r)\,,$$
$$\zeta_{12}\approx\kappa\hat{\zeta}_{12}^{(1)}(r)\,,\quad \zeta_{21}\approx\kappa^2\hat{\zeta}_{21}^{(2)}(r)\,. \tag{18}$$

In $\hat{\zeta}_{ij}^{(k)}$, the superscript k stands for order of perturbation, and i labels the Fourier mode with $j=1$ and 2 being corresponding to $\cos\mathrm{i}\theta$ and $\sin\mathrm{i}\theta$ respectively.

Substituting (17) and (18) into (16), we get the first two terms $\boldsymbol{A}_m$ and $\boldsymbol{A}_{\parallel d}$ as

$$\boldsymbol{A}_\parallel(\boldsymbol{x})=\boldsymbol{A}_m(\boldsymbol{x})+\boldsymbol{A}_{\parallel d}(\boldsymbol{x})+\cdots\,, \tag{19}$$

where

$$\boldsymbol{A}_m(\boldsymbol{x}) = \frac{\Gamma}{4\pi}\int \frac{\boldsymbol{t}(s)}{|\boldsymbol{x}-\boldsymbol{X}(s)|}\mathrm{d}s\,,$$

$$\boldsymbol{A}_{\parallel d}(\boldsymbol{x}) = -\frac{1}{16\pi}\left[2\pi\int_0^\infty r^3\zeta^{(0)}\mathrm{d}r\right]\int \frac{\kappa_s\boldsymbol{n}+\kappa\tau\boldsymbol{b}}{|\boldsymbol{x}-\boldsymbol{X}(s)|}\mathrm{d}s$$

$$-\frac{d^{(1)}}{2}\int \mathrm{d}s\big[\kappa(\boldsymbol{n}\cdot\nabla)+\kappa^2\big]\frac{\boldsymbol{t}}{|\boldsymbol{x}-\boldsymbol{X}(s)|}\,, \quad (20)$$

with

$$d^{(1)} = \frac{1}{4\pi}\left\{\left[2\pi\int_0^\infty r^2\hat{\zeta}_{11}^{(1)}\mathrm{d}r\right]-\frac{1}{2}\left[2\pi\int_0^\infty r^3\zeta^{(0)}\mathrm{d}r\right]\right\}, \quad (21)$$

being the strength of dipole.

The first term $\boldsymbol{A}_m$ in (19) pertains to a flow field induced by a curved vortex line of infinitesimal thickness, and is called the '*monopole field*'. The correction term $\boldsymbol{A}_{\parallel d}$ corresponds to a part of the flow field induced by *a line of dipoles*, based at the vortex centerline, with their axes oriented in the binormal direction. The origin of dipole field is attributable to the curvature effect; by bending a vortex tube, the vortex lines on the convex side are stretched, while those on the concave side are contracted, producing effectively a vortex pair [6].

The components of vorticity perpendicular to $\boldsymbol{t}$ make its appearance at $O(\epsilon^2)$. In view of (8), the second-order terms $\omega_r^{(2)}$ and $\omega_\theta^{(2)}$ are expressible as

$$\omega_r^{(2)} = \frac{\zeta^{(0)}}{v^{(0)}}\hat{\psi}_{11}^{(1)}(\kappa_s\cos\varphi+\kappa\tau\sin\varphi)\,, \quad (22)$$

$$\omega_\theta^{(2)} = \frac{r\zeta^{(0)}}{v^{(0)}}\left[\left(\frac{2}{r}-\frac{\zeta^{(0)}}{v^{(0)}}\right)\hat{\psi}_{11}^{(1)}+\frac{\partial\hat{\psi}_{11}^{(1)}}{\partial r}-rv^{(0)}\right]\times(\kappa\tau\cos\varphi-\kappa_s\sin\varphi)\,, \quad (23)$$

where $\hat{\psi}_{11}^{(1)}$ will be determined as (29) in Section 4.

The vector potential $\boldsymbol{A}_\perp$ associated with the transversal vorticity is, to $O(\epsilon^2)$,

$$\boldsymbol{A}_\perp(\boldsymbol{x}) \approx \frac{1}{4\pi}\int\frac{\mathrm{d}s}{|\boldsymbol{x}-\boldsymbol{X}(s)|}\left[\iint(\omega_r\boldsymbol{e}_r+\omega_\theta\boldsymbol{e}_\theta)\mathrm{d}\tilde{x}\mathrm{d}\tilde{y}\right]. \quad (24)$$

Substitution from (22) and (23) yields

$$\boldsymbol{A}_\perp \approx \frac{1}{4}\left[\int_0^\infty r^2\hat{\zeta}_{11}^{(1)}\mathrm{d}r\right]\int\frac{\kappa_s(s)\boldsymbol{n}(s)+\kappa(s)\tau(s)\boldsymbol{b}(s)}{|\boldsymbol{x}-\boldsymbol{X}(s)|}\mathrm{d}s\,, \quad (25)$$

the *dipole field* originating from the transversal vorticity.

Collecting (20) and (25), we have

$$\boldsymbol{A} \approx \frac{\Gamma}{4\pi}\int\frac{\boldsymbol{t}}{|\boldsymbol{x}-\boldsymbol{X}|}\mathrm{d}s-\frac{d^{(1)}}{2}\int\frac{\kappa\boldsymbol{b}\times(\boldsymbol{x}-\boldsymbol{X})}{|\boldsymbol{x}-\boldsymbol{X}|^3}\mathrm{d}s\,. \quad (26)$$

The tangential component ψ defined by (12) is evaluated near the core ($\sigma_0 \ll r \ll R_0$), which in turn supplies the matching condition on the inner solution. Retaining only the terms with $\log(L/r)$, in the spirit of LIA, and the dipole term, for clarity, we have

$$\psi(\boldsymbol{x}) = d^{(1)}\kappa\frac{\cos\varphi}{r}+\log\left(\frac{L}{r}\right)\left\{\frac{\Gamma}{2\pi}\Big(1-\frac{\kappa}{2}r\cos\varphi\Big)\right.$$
$$+\frac{1}{16\pi}\kappa^2\left[\Gamma\Big(1-\frac{1}{2}\cos 2\varphi\Big)r^2-2d^{(1)}\right]$$
$$+\frac{\Gamma}{32\pi}\left[\Big(\frac{3\kappa^3}{4}-\kappa_{ss}+\kappa\tau^2\Big)\cos\varphi-\frac{\kappa^3}{4}\cos 3\varphi\right.$$
$$\left.-(2\kappa_s\tau+\kappa\tau)\sin\varphi\right]r^3+\frac{d^{(1)}}{2}\left[\Big(-\frac{\kappa^3}{4}+\kappa_{ss}\right.$$
$$\left.\left.-\kappa\tau^2\Big)\cos\varphi+(2\kappa_s\tau+\kappa\tau_s)\sin\varphi\right]r\right\}+\cdots\,. \quad (27)$$

4 Inner solution and filament equation

The inner solution is addressed by solving the Euler equations in the moving coordinates. We introduce the dimensionless variables; the radial distance r is normalized by σ_0, the core radius, time by R_0^2/Γ, the relative velocity by Γ/σ_0 and the centerline velocity by Γ/R_0. With this, we write down dimensionless form of the Euler equations and their curl, viewed from the moving coordinates (r,θ,ξ), along with the subsidiary relation that links ψ to ζ.

The solution at $O(\epsilon)$ is well-known [9–11]:

$$\psi^{(1)} = \Big[\kappa\hat{\psi}_{11}^{(1)}-\big(\dot{\boldsymbol{X}}^{(0)}\cdot\boldsymbol{b}\big)r\Big]\cos\varphi, \quad (28)$$

where

$$\hat{\psi}_{11}^{(1)} = v^{(0)}\left\{\frac{r^2}{2}+\int_0^r\frac{\mathrm{d}r'}{r'[v^{(0)}(r')]^2}\int_0^{r'}r''\left[v^{(0)}(r'')\right]^2\mathrm{d}r''\right\}+c_{11}^{(1)}v^{(0)}, \quad (29)$$

and $c_{11}^{(1)}$ is a constant bearing with the freedom of shifting the local origin $r=0$ of the moving frame, in the $\boldsymbol{n}$-direction, within an accuracy of $O(\epsilon)$ [6]. The matching condition (27) at $O(\epsilon)$ then demands the LIA (1) for $\dot{\boldsymbol{X}}^{(0)}$. The vorticity at $O(\epsilon)$ is calculable through

$$\zeta^{(1)} = -\kappa\left(a\hat{\psi}_{11}^{(1)}+r\zeta^{(0)}\right)\cos\varphi\,. \quad (30)$$

Fortunately an explicit form of $p^{(1)}$ is available by integrating the transversal components of the Euler equations:

$$p^{(1)} = \kappa\left[v^{(0)}\frac{\partial\hat{\psi}_{11}^{(1)}}{\partial r}-\zeta^{(0)}\hat{\psi}_{11}^{(1)}-r(v^{(0)})^2\right]\cos\varphi\,. \quad (31)$$

The gradient of $p^{(1)}$, in turn, drives axial flow at $O(\epsilon^2)$. Discarding the irrelevant terms from the $\boldsymbol{t}$-component of the Euler equations, we obtain

$$-v^{(0)}(\boldsymbol{e}_\theta \cdot \dot{\boldsymbol{t}}^{(0)}) + \frac{v^{(0)}}{r}\frac{\partial w^{(2)}}{\partial \theta} = -\frac{1}{\eta}\frac{\partial p^{(1)}}{\partial \xi} \cdot \qquad (32)$$

A derivative in t of (1) gives

$$\boldsymbol{e}_\theta \cdot \dot{\boldsymbol{t}}^{(0)} = C(\kappa_s \cos\varphi + \kappa\tau \sin\varphi)\,, \qquad (33)$$

and (32) admits a compact form of the solution:

$$w^{(2)} = \left\{ -C\kappa + \frac{\partial \hat{\psi}_{11}^{(1)}}{\partial r} - \frac{\zeta^{(0)}}{v^{(0)}}\hat{\psi}_{11}^{(1)} - r v^{(0)} \right\} r \times (\kappa\tau\cos\varphi - \kappa_s \sin\varphi)\,. \qquad (34)$$

We observe from (34) that torsion and arcwise variation of curvature are vital for the presence of pressure gradient and thus of axial velocity at $O(\epsilon^2)$. The streamfunction $\psi^{(2)}$ at $O(\epsilon^2)$ for flow in the transversal plane is built in parallel with the case of a circular vortex ring [6].

We are now prepared to make headway to third order. The terms proportional to κ^3 in $\dot{\boldsymbol{X}}^{(2)}$ are in complete agreement with those for a circular vortex ring [6]. Hence it suffices to concentrate our attention on the terms tied with torsion and non-constancy of curvature. Retaining only the terms associated with these three-dimensional effects in the vorticity equation at $O(\epsilon^3)$, we are left with

$$\dot{\zeta}^{(1)} - \left(\dot{\boldsymbol{e}}_r^{(0)} \cdot \boldsymbol{e}_\theta\right)\frac{\partial \zeta^{(1)}}{\partial \theta} + \frac{v^{(0)}}{r}\frac{\partial \zeta^{(3)}}{\partial \theta} + u^{(3)}\frac{\partial \zeta^{(0)}}{\partial r} + \cdots = \frac{\zeta^{(0)}}{\eta}\frac{\partial w^{(2)}}{\partial \xi} + \left(\zeta^{(1)} + \kappa r \cos\varphi\,\zeta^{(0)}\right)\frac{\boldsymbol{t}}{\eta}\cdot\frac{\partial \dot{\boldsymbol{X}}^{(0)}}{\partial \xi} + \cdots\,, \qquad (35)$$

the last two terms of which vanish because of (1). The third-order velocity $\dot{\boldsymbol{X}}^{(2)}$ under question is included in $\zeta^{(3)}$ and $u^{(3)}$. Relevant to the traveling speed is the terms proportional to $\cos\varphi$ and $\sin\varphi$, the dipole components.

The first term $\dot{\zeta}^{(1)}$ is obtained from (30) as

$$\dot{\zeta}^{(1)} = -\left(a\hat{\psi}_{11}^{(1)} + r\zeta^{(0)}\right)\left(\dot{\kappa}^{(0)}\cos\varphi + \kappa \dot{T}^{(0)}\sin\varphi\right)\,, \qquad (36)$$

where

$$T(\xi, t) = \int_0^{s(\xi,t)} \tau(s', t)\,\mathrm{d}s'\,, \qquad (37)$$

and $\dot{\kappa}^{(0)}$ and $\dot{T}^{(0)}$ are substituted from the intrinsic form of (1) [1]. Likewise we have

$$\dot{\boldsymbol{e}}_r^{(0)} \cdot \boldsymbol{e}_\theta = C\left(\frac{\kappa_{ss}}{\kappa} - \tau^2\right) - \dot{T}^{(0)}\,. \qquad (38)$$

With these, (35) is integrated for $\zeta^{(3)}$ and is coupled to ζ–ψ relation at $O(\epsilon^3)$. Imposition of the matching condition (27) eventually gives rise to the third-order correction $\dot{\boldsymbol{X}}^{(2)}$. Combining with (1), we arrive at a higher-order extension expressed, in dimensional variables, as

$$\boldsymbol{X}_t = C\left\{\kappa\boldsymbol{b} + c_1\kappa^3\boldsymbol{b} + c_2\Big[(2\kappa_s\tau + \kappa\tau_s)\boldsymbol{n} + (\kappa\tau^2 - \kappa_{ss})\boldsymbol{b} + \kappa^2\tau\boldsymbol{t}\Big]\right\}, \qquad (39)$$

where

$$c_1 = \frac{2\pi d^{(1)}}{\Gamma}\,, \quad c_2 = \frac{\pi}{\Gamma}\int_0^\infty \zeta^{(0)} r^3 \mathrm{d}r\,. \qquad (40)$$

The Hasimoto map (2) transforms (39) into

$$i\phi_t + C\left(\phi_{ss} + \frac{1}{2}|\phi|^2\phi\right) + A(t)\phi - Cc_2\left\{\phi_{ssss} + \frac{3}{2}\left(|\phi|^2\phi_{ss} + \phi_s^2\bar{\phi}\right) + \left(\frac{3}{8}|\phi|^4 + \frac{1}{2}\frac{\partial^2}{\partial s^2}|\phi|^2\right)\phi\right\} + C\left(c_1 + \frac{c_2}{2}\right)\left\{\frac{\partial^2}{\partial s^2}(|\phi|^2\phi) + \frac{3}{4}|\phi|^4\phi\right\} = 0, \qquad (41)$$

where $A(t)$ is an arbitrary function of t. Interestingly equivalent equations have been realized in the context of biquadratic Heisenberg spin chain [12]. The simple form (40) of c_1 is derived by reducing further the formula for a vortex ring [6]. Its derivation will be reported elsewhere.

We see that the third-order correction terms closely resemble $\boldsymbol{V}^{(3)}$. The special case of $c_1 = -c_2/2$ attains integrability. Remember that the dipole strength $d^{(1)}$ and therefore c_1 are sensitive to location of the origin $r = 0$ of the moving coordinates [6]; by a displacement of origin in the $\boldsymbol{n}$-direction by ϵx_0, measured in the inner length-scale,

$$d^{(1)} \to d^{(1)} - x_0/2\pi\,, \quad c_1 \to c_1 - x_0\,. \qquad (42)$$

It is confirmed that c_1 is adjustable so as to be coincident with $\boldsymbol{V}^{(3)}$, and that the local origin for this case is indeed contained inside the core. We conclude that there is an *integrable line* that obeys a summation of the first and the third terms of the LIH.

This fact is illustrated with a specific vorticity distribution at $O(\epsilon^0)$ of constant vorticity in the circular core, that is, the *Rankine vortex*. The azimuthal velocity at $O(\epsilon^0)$ takes, in dimensionless variables,

$$v^{(0)} = \frac{r}{2\pi} \text{ for } r \le 1\,; \quad v^{(0)} = \frac{1}{2\pi r} \text{ for } r > 1\,. \qquad (43)$$

Choosing $c_{11}^{(1)} = -5/8$ amounts to placing the local origin $r = 0$ at the center of core circle, and, from (21), $d^{(1)} = -3/16\pi$. In this case $x_0 = 0$. For a general value of x_0, the third-order terms of (39) become

$$\frac{C}{4}\left\{(2\kappa_s\tau+\kappa\tau_s)\boldsymbol{n}+\left[\kappa\tau^2-\kappa_{ss}-\left(\frac{3}{2}+4x_0\right)\kappa^3\right]\boldsymbol{b}+\kappa^2\tau\boldsymbol{t}\right\}. \tag{44}$$

Recall that the choice of $x_0 = -5/8$ corresponds to placing the origin $r = 0$ at the stagnation point relative to the moving frame. Choice of $x_0 = -1/4$, a value between $x_0 = -5/8$ and 0, renders (39) and (41) completely integrable.

We have extended the matched asymptotic expansions, to third order, for the motion of a vortex filament. This amounts to taking account of finite thickness of the core. The preservation of integrability to third order is indicative of structural stability of the Hasimoto soliton.

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